Math 53, Discussion 105

Wednesday, April 29, 2020 9:19 AM

Review of chapter 16 integrals

For each of the following, write a definition, formula, interpretation, and example

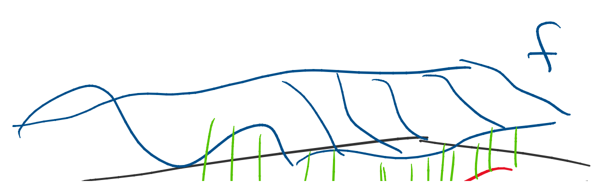
1) Path integral of a scalar function with respect to arc length

Def $$\int_C f\, ds \equiv \int_a^b f(r(t))\, \|r'(t)\|\, dt = \int_a^b \overbrace{f(r(t))}^{d} \sqrt{(x')^2+(y')^2+(z')^2}\, dt$$

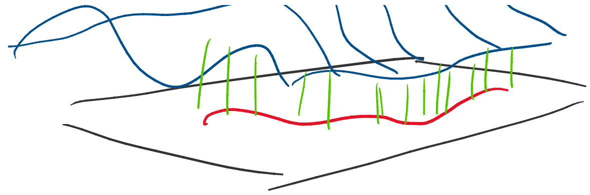
↑ little s

where $r(t) = (x(t), y(t), z(t))$ param of curve C

interpretation: if f is heat in 2d, then this integral is the average heat of f along a curve c, times the length of the curve



the average heat of f along a curve c, times the length of the curve



$$\int F \cdot dr$$

$$\int P\,dx + Q\,dy + R\,dz$$

1) Path integral of a scalar function with respect to x or y

Def $\displaystyle\int_C f\,dx = \int_a^b f(r(t))\,x'(t)\,dt$

with $r(t) = (x(t), y(t))$ Param of C

interpretation:

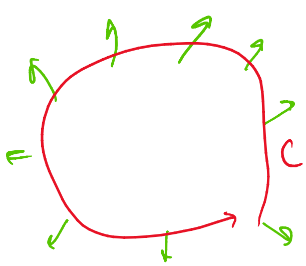
1) Path integral of a vector function

Def $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(r(t)) \cdot r'(t)\,dt = \int_a^b P x'\,dt + Q y'\,dt + R z'\,dt$

$r'(t) = (x', y', z')$ $\qquad F(r(t)) = (P(t), Q(t), R(t))$

Interpretation: go along curve, adding up the amount of the curve that is in the direction of the vector field

(note: if the path at every point is normal to the vector field, then the path integral is zero)



Green = v.f

$\Rightarrow \displaystyle\int_C F \cdot dr = 0$

1) Fundamental theorem of line integrals

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Def: $\int_C \nabla f \cdot d\vec{r} = f(r(b)) - f(r(a))$

with r param. our path

No interpretation (that I can think of), but this is a tool to compute line integrals, given your integrand is conservative (ie gradient of a scalar function)

Remember, vector is conservative can be checked 3 ways (1) curl = 0, (2) path integral over loop is zero (3) for vector field over R^2, check mixed partials $\frac{\partial Q}{\partial x} \stackrel{?}{=} \frac{\partial P}{\partial y}$

1) Greens theorem

Def $\underbrace{\int_C P\,dx + Q\,dy}_{\int_C F\cdot d\vec{r}} = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx$

for closed curve C (positively oriented) & with interior A

Note: lower dimensional version of Stoke's theorem. This is a useful tool to compute path integrals over loops.

1) Curl

$\nabla \times F = \begin{pmatrix}\partial_x \\ \partial_y\end{pmatrix} \times \begin{pmatrix}P \\ Q\end{pmatrix}$

1) Curl

$$\nabla \times F = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$$

1) Divergence

$$\nabla \cdot F = \partial_x P + \partial_y Q + \partial_z R$$

$$\frac{1}{(b-a)} \int_a^b f \, dx = \cancel{(b-a)} (\text{avg } f)$$

1) Surface integral of scalar function

Def $$\iint_S f \, dS = \iint_D f(r(u,v)) \, \|r_u \times r_v\| \, du \, dv$$

with r param of surface S with domain D $\quad r = \langle x(u,v), y(u,v), z(u,v) \rangle$

Interpretation: if S is a surface, and f is temperature, then the surface integral is the average temperature on the surface, times the area of the surface

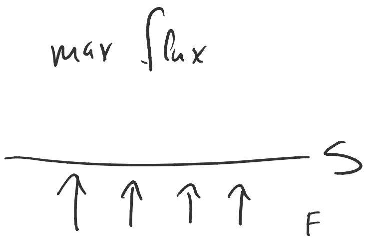
1) Surface integral of vector function

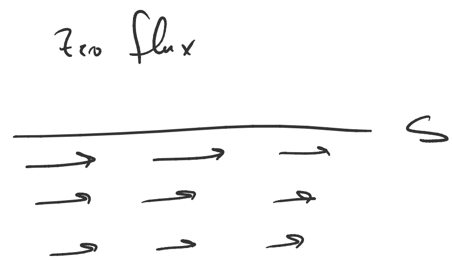
<u>Def</u> $$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dS$$ with $\vec{n}$ the unit vector of S

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, du \, dv$$

with r param of S with domain D

Interpretation: the flux (the amount of stuff that F represents flowing) of F through the surface S





1) Stoke's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

C is loop & S is <u>any</u> surface with boundary C

We use this to compute path integrals of vector fields over loops, if their curl somehow is simple

1) Divergence theorem

$$\underbrace{\iint_S \vec{F} \cdot d\vec{S}}_{\text{flux}} = \iiint_E \nabla \cdot \vec{F} \, dV$$

S = surface closed with interior E